Coherent states of Gompertzian growth

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(Received 16 April 2003; published 25 August 2003)

The origin of the Gompertz function $G(t)=G_0 e^{b/a(1-e^{-at})}$ widely applied to fit the biological and medical data, particularly growth of organisms, organs, and tumors is analyzed. It is shown that this function is a solution of a time-dependent counterpart of the Schrödinger equation for the Morse oscillator with anharmonicity constant equal to 1. The coherent states of the Gompertzian systems, which minimize the time-energy uncertainty relation, have been found. These are eigenstates of the annihilation operator identified with the operator of growth, whereas eigenstates of the creation operator represent the Gompertzian states of regression. The coherent formation of the specific growth patterns in the Gompertzian systems appears as a result of the nonlocal long-range cooperation between the microlevel (the individual cell) and the macrolevel (the system as a whole).

DOI: 10.1103/PhysRevE.68.021916 PACS number(s): 87.10.+e, 05.45.Yv, 89.90.+n, 89.75.Fb

I. INTRODUCTION

Evaluation of the possible growth modes or formation patterns of the systems evolving in time is an important task to realize especially for biological and medical sciences. Investigations in the field have concentrated on (i) finding models of growth which satisfactorily fit the experimental data [1–6], (ii) prediction of the growing systems response to external agents [7–10], and (iii) explanation of the basic mechanisms regulating growth [11–13], in particular cooperative behavior and communication channels between subelements of the growing systems [14–17].

Among the different models of growth, the Gompertz function [18]

$$G(t)=G_0 e^{b/a(1-e^{-at})} \tag{1}$$

has been most broadly and successfully applied to fit the experimental data [19–25], particularly the growth of organisms, organs, tissues, and tumors. Here a is retardation constant; b denotes the initial growth or regression rate constant: the sign of b indicates whether the system grows $(+)$ or regresses $(-)$. The constant $G_0=G(t=0)$ stands for the initial characteristic of the system, for instance, the initial mass, volume, diameter, or number of proliferating cells. An interesting feature of function (1) is that it properly describes growth of the macrosystems composed of a large initial number of proliferating cells ($G_0=10^3$–10^5) [25] as well as microsystems composed of a single cell ($G_0=1$) [24]. In this work we investigate the Gompertzian systems in which $G_0=1$ for the sake of interpretative simplicity

The Gompertz function (1) belongs to the wide class of sigmoidal (S-shaped) functions [3] and describes exponential growth, which then is exponentially retarded and saturated as time continues. The Gompertzian growth is a result of two classes of competitive processes: the first process stimulates growth and the second constrains growth at the saturation stage. The upper limit of the Gompertzian growth is characterized by the asymptote

$$G_\infty=G_0 e^{b/a}, \tag{2}$$

whereas the inflection point is fixed at

$$t_i=-\frac{1}{a}\ln\left(\frac{a}{b}\right), \quad G(t_i)=G_0 e^{(b-a)/a}. \tag{3}$$

The Gompertz function (1) is a solution of the temporal first-order differential equation $(b>0)$ [18]

$$\frac{dG(t)}{dt}-be^{-at}G(t)=0, \tag{4}$$

whereas the regression solution $G^\dagger(t)$ satisfies

$$\frac{dG^\dagger(t)}{dt}+be^{-at}G^\dagger(t)=0. \tag{5}$$

Many attempts have been undertaken to interpret function (1) and Eqs. (4) and (5) in biological [26], mathematical [27], or thermodynamical terms [28]. In this work we interpret the growth of the Gompertzian systems in terms of the time-dependent counterpart of the coherent states of the Morse oscillator. Such states minimize the time-energy uncertainty relation and are eigenstates of the annihilation operator identified with the operator of the Gompertzian growth. We derive also the second-order differential equation governing the Gompertzian growth, which includes Eqs. (4) and (5) as the special cases. This equation is a time-dependent counterpart of the Schrödinger equation for the Morse oscillator with anharmonicity constant equal to 1.

II. GOMPERTZ-MORSE EQUATION

Differentiating function (1) twice with respect to the time coordinate and taking advantage of Eq. (4), we get the second-order differential equation

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$$-\frac{d^2G(t)}{dt^2}-bae^{-at}G(t)+b^2e^{-2at}G(t)=0, \quad (6)$$

which can alternatively be written as

$$-\frac{d^2G(t)}{dt^2}+\frac{a^2}{4}\left(1-\frac{2b}{a}e^{-at}\right)^2G(t)=\frac{a^2}{4}G(t). \quad (7)$$

The function

$$V(t)=\frac{a^2}{4}\left(1-\frac{2b}{a}e^{-at}\right)^2 \quad (8)$$

appearing in Eq. (7) has a minimum equal to zero for

$$t_e=-\frac{1}{a}\ln\left(\frac{a}{2b}\right), \quad (9)$$

hence, Eq. (7) can be rewritten in the form

$$-\frac{1}{2}\frac{d^2G(\tau)}{d\tau^2}+\frac{1}{4}(1-e^{-\sqrt{2}\tau})^2G(\tau)=\frac{1}{4}G(\tau) \quad (10)$$

in which

$$G(\tau)=G_\infty e^{-(1/2)e^{-\sqrt{2}\tau}} \quad (11)$$

and

$$\tau=(\sqrt{2})^{-1}a(t-t_e). \quad (12)$$

The second term in Eq. (10) consists of the time-dependent counterpart of the potential

$$V(x)=D_e(1-e^{-ax})^2 \quad (13)$$

introduced first by Morse [29] to describe anharmonic vibrations in diatomic systems. Here D_e is the dissociation energy of the system, whereas x denotes the displacement of the atom from the equilibrium position.

To explain what Eq. (10) means let us consider the Schrödinger equation for the Morse oscillator

$$-\frac{\hbar^2}{2m_0}\frac{d^2\psi(x)}{dx^2}+D_e(1-e^{-ax})^2\psi(x)=E\psi(x) \quad (14)$$

which transformed to dimensionless coordinate

$$q=(m_0\omega_e/\hbar)^{1/2}x \quad (15)$$

gives [30]

$$-\frac{1}{2}\frac{d^2\psi(q)}{dq^2}+\frac{1}{4x_e}(1-e^{-\sqrt{2x_e}q})^2\psi(q)=\frac{E}{\hbar\omega_e}\psi(q). \quad (16)$$

Here

$$\omega_e=a\sqrt{\frac{2D_e}{m_0}} \quad (17)$$

is the vibrational frequency defined by the reduced mass m_0 of the diatomic system, x_e is anharmonicity constant

$$x_e=\frac{\hbar\omega_e}{4D_e} \quad (18)$$

defined by Planck's constant $h=\hbar 2\pi$.

The Morse procedure [29] applied to Eq. (16) provides eigenvalues

$$E_v=\hbar\omega_e\left[\left(v+\frac{1}{2}\right)-\left(v+\frac{1}{2}\right)^2x_e\right], \quad v=0,1,2\ \dots, \quad (19)$$

and ground-state ($v=0$) eigenfunction

$$\psi_0(q)=e^{-1/2x_e e^{-\sqrt{2x_e}q}}e^{-1/\sqrt{2x_e}(1-x_e)q}; \quad (20)$$

consequently Eq. (16) can be specified in the form [30]

$$-\frac{1}{2}\frac{d^2\psi(q)}{dq^2}+\frac{1}{4x_e}(1-e^{-\sqrt{2x_e}q})^2\psi(q) =\left[\left(v+\frac{1}{2}\right)-\left(v+\frac{1}{2}\right)^2x_e\right]\psi(q). \quad (21)$$

It becomes apparent that the correspondences

$$q\to\tau, \quad x_e\to 1, \quad v\to 0 \quad (22)$$

transform Eq. (21) into Eq. (10) and function (20) into Eq. (11). Hence, the Gompertz function (1) can be viewed as the ground state ($v=0$) solution of the time-dependent counterpart of the Schrödinger equation for the Morse oscillator with anharmonicity constant $x_e=1$.

The question emerges: does Eq. (7) have solutions corresponding to the Morse solutions for $v=1,2\dots$? To answer this question let us rewrite Eq. (7) in the form

$$-\frac{d^2G(t)}{dt^2}+\frac{a^2}{4}\left(1-\frac{2b}{a}e^{-at}\right)^2G(t)=WG(t), \quad (23)$$

which conforms to the standard eigenvalue problem $\hat{W}G(t)=WG(t)$. Then applying the variable

$$y=e^{-\tau'}, \quad \tau'=a(t-t_e), \quad (24)$$

Eq. (23) is transformed to

$$\frac{d^2G(y)}{dy^2}+\frac{1}{y}\frac{dG(y)}{dy}+\left(-\frac{1}{4}+\frac{1}{2y}+\frac{W-a^2/4}{a^2y^2}\right)G(y)=0. \quad (25)$$

Introducing

$$G(y)=e^{-y/2}y^{b/2}f(y), \quad \frac{W-a^2/4}{a^2}=-\frac{b^2}{4} \quad (26)$$

into Eq. (25), one gets

$$y\frac{d^2f(y)}{dy^2}+(b+1-y)\frac{df(y)}{dy}-\frac{b}{2}f(y)=0. \tag{27}$$

The solutions of Eq. (27) can be given by the generalized Laguerre polynomials [29] provided that the last term takes an integer value $-b/2=v$ (v is an integer number). The relation $b=-2v$ together with Eq. (26) provide eigenvalues of Eq. (23)

$$W_v=a^2\left(\frac{1}{4}-v^2\right),\quad v=0,1,2\ \ldots, \tag{28}$$

which for $v=0$ yields the eigenvalue $W_0=a^2/4$ of Eq. (7).

The mathematical functions describing Gompertzian growth or regression should be continuous, single valued, and finite. In this respect these resemble quantal functions of the Q class. Since $G(y)$ defined by Eq. (26) with $y=e^{-\tau'}$ given by Eq. (24) should take finite values for $t\in\langle 0,\infty\rangle$, parameter $b=-2v$ cannot be negative. This criterion is satisfied only for $v=0$; consequently Eq. (23) has only one fundamental (ground-state) eigenmode characterized by the Gompertz function (1), which corresponds to the eigenvalue $W_0=a^2/4$. The later quantity is also the limiting value

$$W_0=\lim_{t\to\infty}\frac{a^2}{4}\left(1-\frac{2b}{a}e^{-at}\right)^2=\frac{a^2}{4} \tag{29}$$

of the time-dependent Morse function (8). We conclude that the Gompertzian growth has only one eigenmode represented by function (1). This eigenmode corresponds to the eigenvalue $W_0=a^2/4$ being an asymptote (for $t\to\infty$) of the Morse function (8). Hence, in the Gompertzian systems the transport of mass (or mass flux) is not of oscillatory type, but takes place only in the direction consistent with the arrow of time (see Fig. 1).

III. GROWTH AND REGRESSION OPERATORS

Introducing two operators

$$\hat{A}=\frac{1}{\sqrt{2}}\left[\frac{d}{d\tau}+\frac{1}{\sqrt{2}}(1-e^{-\sqrt{2}\tau})-\frac{1}{\sqrt{2}}\right], \tag{30}$$

$$\hat{A}^{\dagger}=\frac{1}{\sqrt{2}}\left[-\frac{d}{d\tau}+\frac{1}{\sqrt{2}}(1-e^{-\sqrt{2}\tau})-\frac{1}{\sqrt{2}}\right], \tag{31}$$

the Gompertz-Morse equation (10) can be rearranged into factorized form

$$A^{\dagger}AG(\tau)=0. \tag{32}$$

It is easy to verify that the Gompertz function (11) is a solution of the equation

$$\hat{A}G(\tau)=\frac{1}{\sqrt{2}}\left[\frac{d}{d\tau}-\frac{1}{\sqrt{2}}e^{-\sqrt{2}\tau}\right]G_{\infty}e^{-(1/2)e^{-\sqrt{2}\tau}}=0, \tag{33}$$

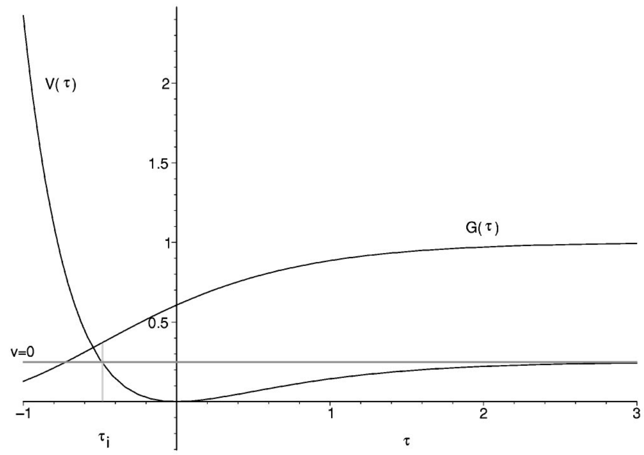

FIG. 1. Plot of the Gompertz function $G(\tau)=G_{\infty}e^{-(1/2)e^{-\sqrt{2}\tau}}$ and Morse function $V(\tau)=\frac{1}{4}(1-e^{-\sqrt{2}\tau})^2$ (dimensionless τ coordinate and $G_{\infty}=1$ are used). The horizontal line represents the ground-state ($v=0$) eigenvalue $W_0=1/4$ of the Gompertz-Morse equation (10), whereas $\tau_i=-\ln(2)/\sqrt{2}$ is the inflection point of the Gompertz curve.

which corresponds to the Gompertz equation of growth (4). Hence, we identify $\hat{A}$ with the operator of growth. On the other hand, operator $\hat{A}^{\dagger}$ produces the regression equation (5)

$$\hat{A}^{\dagger}G^{\dagger}(\tau)=\frac{1}{\sqrt{2}}\left[-\frac{d}{d\tau}-\frac{1}{\sqrt{2}}e^{-\sqrt{2}\tau}\right]G_{\infty}^{\dagger}e^{+(1/2)e^{-\sqrt{2}\tau}}=0, \tag{34}$$

hence, we identify $\hat{A}^{\dagger}$ with the regression operator. It should be pointed out that the change in sign of the regression function $G^{\dagger}(\tau)$ appearing in Eq. (34) is not produced by complex conjugation ($a,b\in\mathcal{R}$, are real numbers) but rather by the relation

$$\psi(\tau)^{\dagger}=\psi(\tau)^{-1} \tag{35}$$

valid for $\psi(\tau)\in\mathcal{R}$.

The growth equation (33) takes a simple form in the ket-bra notation

$$\hat{A}|0\rangle=0, \tag{36}$$

whereas its adjoint

$$(\hat{A}|0\rangle)^{\dagger}=\langle 0|\hat{A}^{\dagger}=0 \tag{37}$$

corresponds to regression equation (34). Here $|0\rangle$ and $\langle 0|$ represent the Gompertz ground states of growth and regression, respectively.

Introducing an analog of the well-known particle number operator $\hat{N}=\hat{A}^{\dagger}\hat{A}$ [31] and a time-dependent function $g(\tau)=e^{-\sqrt{2}\tau}$, we get the following commutation relations:

$$[\hat{A},\hat{A}^{\dagger}]=g(\tau)=-(\hat{A}+\hat{A}^{\dagger}), \tag{38}$$

$$[\hat{N},\hat{A}^{\dagger}]=\hat{A}^{\dagger}g(\tau),\quad [\hat{N},\hat{A}]=-g(\tau)\hat{A}, \tag{39}$$

$$[\hat{A},g(\tau)]=-g(\tau),\quad [\hat{A}^{\dagger},g(\tau)]=g(\tau),$$
$$[\hat{N},g(\tau)]=-g(\tau). \tag{40}$$

The set of operators $\{\hat{N},\hat{A},\hat{A}^{\dagger},g(\tau)\}$ spans the time-dependent analog of the Lie algebra $h(\tau)_4$ in which $g(\tau)=e^{-\sqrt{2}\tau}$ plays a role of the time-dependent "unit" operator. For $g(\tau=0)=1$ commutators (38) and (39) reduce to the well-known relations for harmonic oscillator [31], whereas commutators (40) differ from the corresponding relations [31]. The consistency is retrieved only by making use of the more general relations

$$[\hat{A},g(\tau)]=\frac{1}{\sqrt{2}}\frac{dg(\tau)}{d\tau}=-g(\tau), \tag{41}$$

$$[\hat{A}^{\dagger},g(\tau)]=-\frac{1}{\sqrt{2}}\frac{dg(\tau)}{d\tau}=g(\tau), \tag{42}$$

$$[\hat{N},g(\tau)]=\sqrt{2}\,\frac{dg(\tau)}{d\tau}-[\hat{A},g(\tau)]=-g(\tau) \tag{43}$$

employed to obtain Eq. (40). Now, it becomes clear that for $g(\tau=0)=1$ commutators (41)–(43) are equal to zero as it should be for the harmonic oscillator [31].

IV. GROWTH AND REGRESSION EIGENSTATES

Equations (36) and (37) are time-dependent counterparts of the annihilation and creation relations for the Morse oscillator with anharmonicity constant $x_e=1$. In particular, the action of operator $\hat{A}$ on $G(\tau)$ can be interpreted as annihilation of the Gompertz ground state (fundamental eigenmode) of growth. Exploiting this analogy we found coherent states

$$|\alpha\rangle=G_{\alpha}(\tau)=G_{\infty}e^{\sqrt{2}\alpha\tau}e^{-(1/2)e^{-\sqrt{2}\tau}}=e^{\sqrt{2}\alpha\tau}G(\tau) \tag{44}$$

of the annihilation (growth) operator satisfying [31]

$$\hat{A}|\alpha\rangle=\alpha|\alpha\rangle. \tag{45}$$

From Eq. (45) one gets relation for the creation (regression) states

$$\langle\alpha|\hat{A}^{\dagger}=\langle\alpha|\alpha^{*}, \tag{46}$$

with solutions

$$\langle\alpha|=G^{\dagger}_{\alpha}(\tau)=G^{\dagger}_{\infty}e^{-\sqrt{2}\alpha^{*}\tau}e^{(1/2)e^{-\sqrt{2}\tau}}=e^{-\sqrt{2}\alpha^{*}\tau}G^{\dagger}(\tau). \tag{47}$$

The Gompertz coherent states satisfy unitary relation

$$G_{\alpha}(\tau)^{\dagger}=\{[G_{\alpha}(\tau)]^{*}\}^{-1}, \tag{48}$$

hence, for real eigenvalues $\alpha\in\mathcal{R}$ the Gompertz states of growth are simply related to the reciprocal states of regression (35).

Although the Gompertz coherent states are represented by the finite, single valued, and continuous functions (44), these do not satisfy the boundary conditions usually applied to quantal systems. For example, these may not disappear at infinity. Consequently, normalization of the Gompertz states is not possible in the infinite time intervals. Since the process of biological growth takes place in the finite periods, due to finite time existence of the systems under consideration, we can normalize such states in the finite interval $t\in\langle 0,n\rangle$. Keeping in mind this restriction let us multiply Eq. (45) on the left side by $\langle\alpha|$, whereas Eq. (46) on the right side by $|\alpha\rangle$ getting

$$\langle\alpha|\hat{A}|\alpha\rangle=\alpha\langle\alpha|\alpha\rangle, \tag{49}$$

$$\langle\alpha|\hat{A}^{\dagger}|\alpha\rangle=\alpha^{*}\langle\alpha|\alpha\rangle. \tag{50}$$

For the ground state of growth and regression when $\alpha=0$ (generally $\alpha\in\mathcal{R}$) one gets

$$\langle\alpha|\alpha\rangle=n, \tag{51}$$

whereas for complex $\alpha\in\mathcal{Z}$

$$\langle\alpha|\alpha\rangle=\frac{e^{\sqrt{2}n(\alpha-\alpha^{*})}-1}{\sqrt{2}(\alpha-\alpha^{*})}. \tag{52}$$

In both cases the expected values of the annihilation (creation) operator are generated from the relations

$$\frac{\langle\alpha|\hat{A}|\alpha\rangle}{\langle\alpha|\alpha\rangle}=\alpha,\quad \frac{\langle\alpha|\hat{A}^{\dagger}|\alpha\rangle}{\langle\alpha|\alpha\rangle}=\alpha^{*}. \tag{53}$$

To prove that Eq. (44) represents coherent states normalized according to the above described procedure, let us consider the time-dependent analog of the Nieto-Simmons equation [32]

$$\left[T(\tau)-i\hat{E}\frac{\Delta T}{\Delta E}\right]|\alpha\rangle=\left[\langle\alpha|T(\tau)|\alpha\rangle-i\langle\alpha|\hat{E}|\alpha\rangle\frac{\Delta T}{\Delta E}\right]|\alpha\rangle \tag{54}$$

in which

$$\hat{E}=i\frac{d}{d\tau} \tag{55}$$

is the energy operator expressed in τ coordinate ($\hbar=1$), whereas

$$T(\tau)=-\frac{1}{\sqrt{2}}e^{-\sqrt{2}\tau} \tag{56}$$

is, to within a constant, the time-dependent Morse variable appearing in Eqs. (30) and (31). The remaining terms are defined as follows:

$$\Delta T=\sqrt{\langle\alpha|T(\tau)^2|\alpha\rangle-\langle\alpha|T(\tau)|\alpha\rangle^2}, \tag{57}$$

$$\Delta E=\sqrt{\langle\alpha|\hat{E}^2|\alpha\rangle-\langle\alpha|\hat{E}|\alpha\rangle^2}. \tag{58}$$

Equation (54) simplified to the form

$$\left[T(\tau)+\lambda\frac{d}{d\tau}\right]|\alpha\rangle=\left[\langle\alpha|T(\tau)|\alpha\rangle+\lambda\langle\alpha|\frac{d}{d\tau}|\alpha\rangle\right]|\alpha\rangle, \tag{59}$$

where

$$\lambda=\frac{\Delta T}{\Delta E} \tag{60}$$

has the solutions

$$|\alpha\rangle=G_\infty e^{-(1/2\lambda)e^{-\sqrt{2}\tau}}e^{\sqrt{2}\alpha\tau/\lambda}, \tag{61}$$

including Eq. (44) as a special case ($\lambda=1$), provided that the right-hand side of equation (59) reduces to

$$\left[\langle\alpha|T(\tau)|\alpha\rangle+\lambda\langle\alpha|\frac{d}{d\tau}|\alpha\rangle\right]=\sqrt{2}\alpha. \tag{62}$$

By differentiating function (61) once with respect to τ coordinate, one may calculate

$$\lambda\langle\alpha|\frac{d}{d\tau}|\alpha\rangle=-\langle\alpha|T(\tau)|\alpha\rangle+\sqrt{2}\alpha\langle\alpha|\alpha\rangle, \tag{63}$$

hence, relation (62) is satisfied for $\langle\alpha|\alpha\rangle=1$ normalized according to Eq. (52).

For $\lambda=1$, Eq. (59) reproduces (45), whereas function Eq. (61) reduces to (44). The interpretation of the case $\lambda=1$ is given in the following section.

V. MINIMUM UNCERTAINTY COHERENT STATES

The ordinary (space-dependent) coherent states of microsystems are defined as [31] (i) eigenstates of the annihilation operator, (ii) states that minimize the position-momentum uncertainty relation and (iii) states that arise from the operation of a unitary displacement operator to the ground state of the microsystem. In the preceding section, we proved that the Gompertz coherent states are eigenstates of the annihilation operator identified with the operator of the Gompertzian growth. Here, we prove that such states minimize the time-energy uncertainty relation in which the original τ coordinate is replaced by its exponential form (56).

The space-dependent coherent states of the Morse oscillator minimize the position-momentum uncertainty relation [30]

$$(\Delta Q)^2(\Delta P)^2\geqslant\tfrac{1}{4}\langle\alpha|f(q)|\alpha\rangle^2,\quad [Q(q),\hat{P}]=if(q), \tag{64}$$

in which $\hat{P}=-id/dq$ ($\hbar=1$) is the momentum operator and

$$Q(q)=-\frac{1}{\sqrt{2x_e}}e^{-\sqrt{2x_e}q}+\frac{1}{\sqrt{2x_e}}(1-x_e) \tag{65}$$

is, to within a constant, the space-dependent Morse variable [30], which depends on the q-coordinate defined by Eq. (15).

On the contrary, the time-dependent coherent states (44) should minimize the time-energy uncertainty relation

$$(\Delta T)^2(\Delta E)^2\geqslant\tfrac{1}{4}\langle\alpha|g(\tau)|\alpha\rangle^2,\quad [T(\tau),\hat{E}]=ig(\tau), \tag{66}$$

including the $T(\tau)$ variable and energy operator $\hat{E}$ defined by Eqs. (56) and (55), respectively. It is easy to verify that $Q(x_e=1,q=\tau)=T(\tau)$.

We confirm that states (44) are indeed the minimum uncertainty coherent states by proving that

$$(\Delta T)^2(\Delta E)^2=\tfrac{1}{4}\langle\alpha|g(\tau)|\alpha\rangle^2 \tag{67}$$

for

$$[T(\tau),\hat{E}]=ig(\tau),\quad g(\tau)=e^{-\sqrt{2}\tau}. \tag{68}$$

To this effect let us calculate

$$\langle\alpha|T(\tau)|\alpha\rangle=\frac{1}{\sqrt{2}}\langle\alpha|\hat{A}+\hat{A}^\dagger|\alpha\rangle=\frac{1}{\sqrt{2}}(\alpha+\alpha^*), \tag{69}$$

$$\langle\alpha|\hat{E}|\alpha\rangle=i\frac{1}{\sqrt{2}}\langle\alpha|\hat{A}-\hat{A}^\dagger|\alpha\rangle=i\frac{1}{\sqrt{2}}(\alpha-\alpha^*), \tag{70}$$

$$\begin{aligned}\langle\alpha|T(\tau)^2|\alpha\rangle&=\tfrac{1}{2}\langle\alpha|\hat{A}\hat{A}+2\hat{A}^\dagger\hat{A}+g(\tau)+\hat{A}^\dagger\hat{A}^\dagger|\alpha\rangle\\&=\tfrac{1}{2}[(\alpha+\alpha^*)^2+\langle\alpha|g(\tau)|\alpha\rangle],\end{aligned} \tag{71}$$

$$\begin{aligned}\langle\alpha|\hat{E}^2|\alpha\rangle&=-\tfrac{1}{2}\langle\alpha|\hat{A}\hat{A}-2\hat{A}^\dagger\hat{A}-g(\tau)+\hat{A}^\dagger\hat{A}^\dagger|\alpha\rangle\\&=-\tfrac{1}{2}[(\alpha-\alpha^*)^2-\langle\alpha|g(\tau)|\alpha\rangle].\end{aligned} \tag{72}$$

To obtain relations (71) and (72) we employed commutator (38) given in the form

$$\hat{A}\hat{A}^\dagger=\hat{A}^\dagger\hat{A}+g(\tau). \tag{73}$$

Having calculated Eqs. (69)–(72) we can proceed to evaluate

$$(\Delta T)^2=\langle\alpha|T(\tau)^2|\alpha\rangle-\langle\alpha|T(\tau)|\alpha\rangle^2=\tfrac{1}{2}\langle\alpha|g(\tau)|\alpha\rangle, \tag{74}$$

$$(\Delta E)^2=\langle\alpha|\hat{E}^2|\alpha\rangle-\langle\alpha|\hat{E}|\alpha\rangle^2=\tfrac{1}{2}\langle\alpha|g(\tau)|\alpha\rangle, \tag{75}$$

providing

$$(\Delta T)^2(\Delta E)^2=\tfrac{1}{4}\langle\alpha|g(\tau)|\alpha\rangle^2, \tag{76}$$

in full agreement with Eq. (67). Additionally, from Eqs. (74) and (75) one gets the relation $\Delta T/\Delta E=\lambda=1$ appearing in the Nieto-Simmons formula (54). In this way we prove that Eq. (54) is satisfied by the Gompertz coherent states minimizing the time-energy uncertainty relation (66).

For the ordinary coherent states of the Morse oscillator we have $\Delta Q=\Delta P=\text{const}$ [30] in which $Q(q)$ is the spatial variable defined by Eq. (65). Hence, the coherent states of the Morse oscillator evolve coherently in time being localized on the classical space trajectory. In the case of the Gompertz states (44), we have $\Delta T=\Delta E=\text{const}$ in which

$T(\tau)$ is the temporal variable defined by Eq. (56). In view of this the time-dependent Gompertzian states evolve coherently in space being localized on the classical time trajectory. It becomes apparent that the spatial coherence is an immanent feature of the Gompertzian growth. A generalization of the Schrödinger approach [33] to time evolution of the coherent states of the harmonic oscillator to include spatial evolution of the time-dependent Gompertz coherent states will be presented in separate paper.

VI. CONCLUDING REMARKS

We have derived the temporal second-order differential equation governing growth of the Gompertzian systems. This equation, expressed in dimensionless coordinate, has the form identical to that of the quantal Schrödinger equation for the time-dependent analog of the Morse oscillator with anharmonicity constant equal to 1. The Gompertz-Morse equation has only one finite, single valued, and continuous solution, which corresponds to the fundamental eigenmode of the Gompertzian growth. This eigenmode is represented by the Gompertz function of growth, whereas the associated eigenvalue is equal to the depth of the Morse function at the minimum. The transport of mass in the Gompertzian systems is driven by the time-dependent counterpart of the Morse potential. This process takes place in the direction consistent with the arrow of time and resembles dissociation of an ordinary anharmonic oscillator.

The coherent states of the Gompertzian systems have been derived. These are the eigenstates of the annihilation operator identified with the operator of the Gompertzian growth. Such states evolve coherently in space being localized along the classical time trajectory, hence, the Gompertzian growth is predicted to be coherent in space. We find here a strong analogy to the spatial long-range biocoherence reported by Fröhlich [34] and macroscopic quantal coherence in Bose-Einstein condensates [35–37]. In the Fröhlich model a system of coupled oscillators in a heat bath is supplied with energy at a constant rate. When the rate exceeds a certain mean rate, the oscillators condense into one giant dipole whose subelements are spatially inter-related to each other. This phenomenon features a considerable similarity with the low-temperature condensation of Bose-Einstein gas [34]. The macroscopic quantal coherence is observed in the systems composed of a large number (10^3 for ^{7}Li, 10^6 for ^{87}Rb and 10^7 for ^{23}Na) of trapped cold atoms [35–37]. In this phenomenon, all particles cooperate collectively producing spatiotemporal organization of the multiparticle system, in which all particles share the same quantum state [35–37].

We recall here that according to the Laird result [24] the Gompertz function (1) evaluated for the system of proliferating cells can be extrapolated to one cell. It means that this function properly describes coherent growth of the macrosystem (organism, organ, tumor) as a whole and its subsystems (microsystems) composed of a single. Those subsystems are spatially inter-related and share the same state (mode) of growth as the system as a whole. Such a long-range cooperation enables the system to develop complex patterns in response to external and internal conditions. This response requires self-organization on all levels of the system and cooperative behavior of all its subelements. The local models of cell-to-cell communication [38] via (i) molecular signalling (hormones, cytokines, neurotransmitters, etc.), (ii) extracellular matrix (integrins, microtubules, actin, etc.), or (iii) gap junctions (mediated by ions or current flow) do not explain satisfactorily the observed long-range spatial coherence of the growing systems and decoherence leading to uncontrolled growth and malignancy.

The coherent formation of the specific growth patterns in the Gompertzian systems seems to be a result of the nonlocal long-range cooperation between the microlevel (the individual cells) and the macrolevel (the system as a whole). The nonlocal communication channel enables each cell to obtain information about the state of the system and respond to it adequately. Such nonlocal cooperative self-organization and intricate communication capabilities have been observed in the bacterial colonies by Ben-Jacob [39] and his co-workers [40]. The former include (i) collective production of extracellular "wetting" fluid for movement on hard surfaces [41], (ii) long-range chemical signalling, e.g., quorum sensing [42] and chemotactic signaling [15], and (iii) collective activation and deactivation of genes [16]. Owing to these capabilities, the bacterial colonies develop complex spatiotemporal patterns in response to adverse growth conditions. This process is accomplished via cooperative complexification of the colony through hierarchical self-organized patterning mediated by the information transfer between the individual bacterium (microlevel) and the colony as a whole (macrolevel) [39,40]. Such a long-range communication between microorganisms can be realized through a nonlocal bio-signalling; the appropriate communication model has been developed by Ben-Jacob *et al.* [14]. It should be noted here that growth of the bacterial colonies is well reproduced by the temperature-dependent Gompertz function introduced by Zwietering *et al.* [43]. The nonlinear regression analysis performed on the experimental data obtained for the pathogens *Listeria monocytogenes* and *Yersinia enterocolitica* provided excellent fit of the Gompertz model with observed growth of the bacteria inoculated in chicken meat [44]. On the basis of the results obtained in this work, we conclude that growth of the bacterial colonies characterized by the Gompertz function is coherent in space.

The macroscopic long-range spatial coherence is observed also in highway traffic. Helbing and Huberman [45] reported coherent moving states, which arise from cooperative interactions between vehicles. As the density of vehicles increase, their interactions cause a transition into a highly coherent state in which all vehicles have the same average velocity and a small dispersion around this value. The theoretical predictions of Helbing and Huberman have been confirmed by empirical data obtained from highway traffic in the Netherlands [45].

The Gompertz coherent states seem to be a convenient tool for interpretation of interpret the micro-macro correspondences so far intensively studied only in terms of the ordinary space-dependent coherent states [46]. For example, using the Husimi Q representation [47], one finds that there are precise patterns of classical trajectories in the wave function. The Q distribution follows the classical periodic orbits [48], whereas classical unstable periodic orbits are endowed with ''scars'' in the Q distribution [49].

ACKNOWLEDGMENTS

We are indebted to Dr. P. Waliszewski who brought the Gompertz model to our attention and to Professor M. Wnuk who provided materials indispensable in writing this paper. Special thanks are due to M. Mańkowski for the software support.

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